

# SUBVARIETIES OF THE TETRABLOCK AND VON-NEUMANN'S INEQUALITY

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**ABSTRACT.** We show an interplay between the complex geometry of the tetrablock  $\mathbb{E}$  and the commuting triples of operators having  $\overline{\mathbb{E}}$  as a spectral set. We prove that  $\overline{\mathbb{E}}$  being a 3-dimensional domain does not have any 2-dimensional distinguished variety, every distinguished variety in the tetrablock is one-dimensional and can be represented as

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}, \quad (0.1)$$

where  $A_1, A_2$  are commuting matrices of the same order satisfying  $[A_1^*, A_1] = [A_2^*, A_2]$  and a norm condition. The converse also holds, i.e, a set of the form (0.1) is always a distinguished variety in  $\mathbb{E}$ . We show that for a triple of commuting operators  $\Upsilon = (T_1, T_2, T_3)$  having  $\overline{\mathbb{E}}$  as a spectral set, there is a one-dimensional subvariety  $\Omega_\Upsilon$  of  $\overline{\mathbb{E}}$  depending on  $\Upsilon$  such that von-Neumann's inequality holds, i.e,

$$f(T_1, T_2, T_3) \leq \sup_{(x_1, x_2, x_3) \in \Omega_\Upsilon} |f(x_1, x_2, x_3)|,$$

for any holomorphic polynomial  $f$  in three variables, provided that  $T_3^n \rightarrow 0$  strongly as  $n \rightarrow \infty$ . The variety  $\Omega_\Upsilon$  has been shown to have representation like (0.1), where  $A_1, A_2$  are the unique solutions of the operator equations

$$T_1 - T_2^* T_3 = (I - T_3^* T_3)^{\frac{1}{2}} X_1 (I - T_3^* T_3)^{\frac{1}{2}} \text{ and}$$

$$T_2 - T_1^* T_3 = (I - T_3^* T_3)^{\frac{1}{2}} X_2 (I - T_3^* T_3)^{\frac{1}{2}}.$$

We also show that under certain condition,  $\Omega_\Upsilon$  is a distinguished variety in  $\mathbb{E}$ . We produce an explicit dilation and a concrete functional model for such a triple  $(T_1, T_2, T_3)$  in which the unique operators  $A_1, A_2$  play the main role. Also, we describe a connection of this theory with the distinguished varieties in the bidisc and in the symmetrized bidisc.

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## 1. INTRODUCTION

The tetrablock is a polynomially convex, non-convex and inhomogeneous domain in  $\mathbb{C}^3$  defined by

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zw x_3 \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\}.$$

This domain has attracted a lot of attention of the function theorists, complex geometers and operator theorists over past one decade because of its connection with  $\mu$ -synthesis and  $H^\infty$  control theory ([1, 2, 33, 16, 17, 34, 10, 13, 27]). The *distinguished boundary* of the tetrablock, which is same as the Šilov boundary, was determined in [1] (Theorem 7.1 in [1]) to be the set

$$\begin{aligned} b\mathbb{E} &= \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 = \bar{x}_2 x_3, |x_3| = 1 \text{ and } |x_2| \leq 1\} \\ &= \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : |x_3| = 1\}. \end{aligned} \quad (1.1)$$

Amongst the characterizations given in [1] of the points in  $\mathbb{E}$ , the following is the most elegant one that clarifies the geometric location of  $\mathbb{E}$ . This will be used frequently in all the sections of this paper.

**Theorem 1.1.** *A point  $(x_1, x_2, x_3) \in \mathbb{C}^3$  is in  $\mathbb{E}$  (respectively in  $\overline{\mathbb{E}}$ ) if and only if  $|x_3| < 1$  (respectively  $\leq 1$ ) and there exist  $\beta_1, \beta_2 \in \mathbb{C}$  such that  $|\beta_1| + |\beta_2| < 1$  (respectively  $\leq 1$ ) and  $x_1 = \beta_1 + \bar{\beta}_2 x_3$ ,  $x_2 = \beta_2 + \bar{\beta}_1 x_3$ .*

It is evident from the above theorem that the tetrablock lives inside the tridisc  $\mathbb{D}^3$  and that the topological boundary  $\partial\mathbb{E}$  of the tetrablock is given by

$$\begin{aligned} \partial\mathbb{E} &= \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : |x_3| = 1\} \\ &\cup \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : |x_3| < 1, |\beta_1| + |\beta_2| = 1\} \\ &= b\mathbb{E} \cup \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : |x_3| < 1, |\beta_1| + |\beta_2| = 1\}, \end{aligned}$$

where  $\beta_1, \beta_2$  are as of Theorem 1.1.

A variety  $V_S$  in  $\mathbb{C}^n$ , where  $S$  is a set of polynomials in  $n$ -variables  $z_1, \dots, z_n$ , is a subset of  $\mathbb{C}^n$  defined by

$$V_S = \{(z_1, \dots, z_n) \in \mathbb{C}^n : p(z_1, \dots, z_n) = 0, \text{ for all } p \in S\}.$$

A variety  $W$  in a domain  $G \subseteq \mathbb{C}^n$  is the part of a variety lies inside  $G$ , i.e,  $W = V_S \cap G$  for some set  $S$  of polynomials in  $n$ -variables. A *distinguished*

*variety* in a polynomially convex domain  $G$  is a variety that intersects the topological boundary of  $G$  at its Šilov boundary. See [3, 4, 24, 28] to follow some recent works on distinguished varieties in the bidisc and in the symmetrized bidisc. Therefore, in particular a distinguished variety in the tetrablock is defined in the following way.

**Definition 1.2.** A set  $\Omega \subseteq \mathbb{E}$  is said to be a distinguished variety in the tetrablock if  $\Omega$  is a variety in  $\mathbb{E}$  such that

$$\overline{\Omega} \cap \partial\mathbb{E} = \overline{\Omega} \cap b\overline{\mathbb{E}}. \quad (1.2)$$

We denote by  $\partial\Omega$  the set described in (1.2). It is evident from the definition that a distinguished variety in  $\mathbb{E}$  is a one or two-dimensional variety in  $\mathbb{E}$  that exits the tetrablock through the distinguished boundary  $b\mathbb{E}$ .

The main aim of this paper is to build and explain a connection between the complex geometry of the domain  $\mathbb{E}$  and the triple of commuting operators having  $\overline{\mathbb{E}}$  as a spectral set. The principal source of motivation for us is the seminal paper [3] of Agler and M<sup>c</sup>Carthy. In our first main result Theorem 4.5, we show that no distinguished variety in  $\mathbb{E}$  can be 2-dimensional, all distinguished varieties in  $\mathbb{E}$  have complex dimension one and can be represented as

$$\{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}, \quad (1.3)$$

where  $A_1, A_2$  are commuting matrices of same order such that  $[A_1^*, A_1] = [A_2^*, A_2]$  and that  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} \leq 1$ ,  $\mathbb{T}$  being the unit circle in the complex plane and

$$\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} = \sup_{z \in \mathbb{T}} \|A_1^* + A_2 z\|.$$

Here  $\sigma_T(S_1, \dots, S_n)$  denotes the *Taylor joint spectrum* of a commuting  $n$ -tuple of operators  $(S_1, \dots, S_n)$  which consists of joint eigenvalues of  $S_1, \dots, S_n$  only when they are matrices (see [23] for a detailed proof of this fact). Since only the case  $n = 2$  is used here, we shall have a brief discussion on Taylor joint spectrum of a pair of commuting bounded operators at the beginning of section 2. Also  $[S_1, S_2]$  denotes the commutator  $S_1 S_2 - S_2 S_1$ . Conversely, every subset of the form (1.3) is a distinguished variety in  $\mathbb{E}$  provided that  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} < 1$ . Examples show that a set of this kind may or may not be a distinguished variety if  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} = 1$ . It is surprising that the tetrablock, being a domain of complex dimension 3, does not have a two-dimensional distinguished variety. Thus, the study of the distinguished varieties in  $\mathbb{E}$  leads us to operator theory, in particular to matrix theory.

On the other hand, another main result, Theorem 6.1, shows that if a triple of commuting operators  $\Upsilon = (T_1, T_2, T_3)$ , defined on a Hilbert space  $\mathcal{H}$ , has  $\overline{\mathbb{E}}$  as a spectral set and if the fundamental operators  $A_1, A_2$  of  $\Upsilon$  are commuting matrices satisfying  $[A_1^*, A_1] = [A_2^*, A_2]$ , then there is a one-dimensional subvariety  $\Omega_\Upsilon$  of  $\overline{\mathbb{E}}$  such that  $\Omega_\Upsilon$  is a spectral set for  $\Upsilon$ . The fundamental operators  $A_1, A_2$  are two unique operators associated with such

a commuting triple  $(T_1, T_2, T_3)$  and are explained below. Also  $\omega(T)$  denotes the numerical radius of an operator  $T$ . The one-dimensional subvariety  $\Omega_\Upsilon$  is obtained in terms of the fundamental operators  $A_1, A_2$  in the following way,

$$\Omega_\Upsilon = \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}.$$

Moreover, when  $A_1, A_2$  satisfy the condition  $\omega(A_1 + A_2 z) < 1$ , for all  $z \in \mathbb{T}$ ,  $\Omega_\Upsilon \cap \mathbb{E}$  is a distinguished variety in  $\mathbb{E}$ . So, it is remarkable that for such an operator triple  $\Upsilon$  one can extract a one-dimensional curve from (3-dimensional)  $\mathbb{E}$  on which  $\Upsilon$  lives. Therefore, the study of commuting operator triples that have  $\overline{\mathbb{E}}$  as a spectral set, takes us back to the complex geometry of  $\overline{\mathbb{E}}$ .

**Definition 1.3.** A commuting triple of operators  $(T_1, T_2, T_3)$  that has  $\overline{\mathbb{E}}$  as a spectral set is called an  $\mathbb{E}$ -contraction, i.e, an  $\mathbb{E}$ -contraction is a commuting triple  $(T_1, T_2, T_3)$  such that  $\sigma_T(T_1, T_2, T_3) \in \overline{\mathbb{E}}$  and that for every holomorphic polynomial  $p$  in three variables

$$\|p(T_1, T_2, T_3)\| \leq \sup_{(x_1, x_2, x_3) \in \overline{\mathbb{E}}} |p(x_1, x_2, x_3)| = \|p\|_{\infty, \overline{\mathbb{E}}}.$$

Since the set  $\mathbb{E}$  is contained in the tridisc  $\mathbb{D}^3$  as was shown in [1], an  $\mathbb{E}$ -contraction consists of commuting contractions. Also it is merely mentioned that if  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction then so is the adjoint  $(T_1^*, T_2^*, T_3^*)$ . In [10], Bhattacharyya introduced the study of  $\mathbb{E}$ -contractions by using the efficient machinery so called *fundamental operators*. We mention here that in [10], a triple  $(T_1, T_2, T_3)$  that had  $\overline{\mathbb{E}}$  as a spectral set was called a *tetablock contraction*. But since a notation is always more convenient when writing, we prefer to call them  $\mathbb{E}$ -contractions. It was shown in Theorem 3.5 in [10] that to every  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$ , there are two unique operators  $A_1, A_2$  on  $\mathcal{D}_{T_3}$  such that

$$T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3} \text{ and } T_2 - T_1^* T_3 = D_{T_3} A_2 D_{T_3}.$$

For a contraction  $T$ , we shall always denote by  $D_T$  the positive operator  $(I - T^* T)^{\frac{1}{2}}$  and  $\mathcal{D}_T = \overline{\text{Ran}} D_T$ . An explicit dilation was constructed in [10] for a particular class of  $\mathbb{E}$ -contractions (see Theorem 6.1 in [10]). Since these two operators  $A_1, A_2$  were the key ingredients in that construction, they were named the *fundamental operators* of  $(T_1, T_2, T_3)$ . The fundamental operators always satisfy  $\omega(A_1 + A_2 z) \leq 1$ , for all  $z$  in  $\mathbb{T}$ . For a further reading on  $\mathbb{E}$ -contractions, fundamental operators and their properties, see [13, 14]. Also in a different direction to know about the failure of rational dilation on the tetablock, an interested reader is referred to [27].

Unitaries, isometries and co-isometries are special types of contractions. There are natural analogues of these classes for  $\mathbb{E}$ -contractions in the literature.

**Definition 1.4.** Let  $T_1, T_2, T_3$  be commuting operators on a Hilbert space  $\mathcal{H}$ . We say that  $(T_1, T_2, T_3)$  is

- (i) an  $\mathbb{E}$ -unitary if  $T_1, T_2, T_3$  are normal operators and the joint spectrum  $\sigma_T(T_1, T_2, T_3)$  is contained in  $b\mathbb{E}$  ;
- (ii) an  $\mathbb{E}$ -isometry if there exists a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$  and an  $\mathbb{E}$ -unitary  $(\tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$  on  $\mathcal{K}$  such that  $\mathcal{H}$  is a common invariant subspace of  $T_1, T_2, T_3$  and that  $T_i = \tilde{T}_i|_{\mathcal{H}}$  for  $i = 1, 2, 3$  ;
- (iii) an  $\mathbb{E}$ -co-isometry if  $(T_1^*, T_2^*, T_3^*)$  is an  $\mathbb{E}$ -isometry.

**Definition 1.5.** An  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  is said to be *pure* if  $T_3$  is a pure contraction, i.e,  $T_3^{*n} \rightarrow 0$  strongly as  $n \rightarrow \infty$ . Similarly an  $\mathbb{E}$ -isometry  $(T_1, T_2, T_3)$  is said to be a pure  $\mathbb{E}$ -isometry if  $T_3$  is a pure isometry, i.e, equivalent to a shift operator.

**Definition 1.6.** Let  $(T_1, T_2, T_3)$  be a  $\mathbb{E}$ -contraction on  $\mathcal{H}$ . A commuting triple  $(Q_1, Q_2, V)$  on  $\mathcal{K}$  is said to be an  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, T_3)$  if  $\mathcal{H} \subseteq \mathcal{K}$ ,  $(Q_1, Q_2, V)$  is an  $\mathbb{E}$ -isometry and

$$P_{\mathcal{H}}(Q_1^{m_1} Q_2^{m_2} V^n)|_{\mathcal{H}} = T_1^{m_1} T_2^{m_2} T_3^n, \text{ for all non-negative integers } m_1, m_2, n.$$

Here  $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{H}$  is the orthogonal projection of  $\mathcal{K}$  onto  $\mathcal{H}$ . Moreover, the dilation is called *minimal* if

$$\mathcal{K} = \overline{\text{span}}\{Q_1^{m_1} Q_2^{m_2} V^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N} \cup \{0\}\}.$$

In section 3, we add to the account some operator theory on the tetrablock. In Theorem 3.2, another main result of this paper, we construct an  $\mathbb{E}$ -isometric dilation to a pure  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  whose adjoint has commuting fundamental operators  $A_{1*}, A_{2*}$  such that  $[A_{1*}^*, A_{1*}] = [A_{2*}^*, A_{2*}]$ . This dilation is different from the one established in [10] (Theorem 6.1 in [10]) and here the dilation operators involve the fundamental operators of the adjoint  $(T_1^*, T_2^*, T_3^*)$ . We show further that the dilation is minimal. As a consequence of this dilation, we obtain a functional model in Theorem 3.4 for such pure  $\mathbb{E}$ -contractions in terms of commuting Toeplitz operators on the vectorial Hardy space  $H^2(\mathcal{D}_{T_3^*})$ . Theorem 3.3 describes a set of sufficient conditions under which a triple of commuting contractions  $(T_1, T_2, T_3)$  becomes an  $\mathbb{E}$ -contraction. Indeed, if there are two commuting operators  $A_1, A_2$  that satisfy  $T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3}$  and  $T_2 - T_1^* T_3 = D_{T_3} A_2 D_{T_3}$ , then  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction provided that  $[A_1^*, A_1] = [A_2^*, A_2]$  and  $\omega(A_1 + A_2 z) \leq 1$ , for every  $z$  in the unit circle. Also in Corollary 3.7, we show that a pair of commuting operators  $A_1, A_2$  on a Hilbert space  $E$ , satisfying  $[A_1^*, A_1] = [A_2^*, A_2]$  and  $\omega(A_1 + A_2 z) \leq 1$ , for all  $z$  of unit modulus, are the fundamental operators of an  $\mathbb{E}$ -contraction defined on the vectorial Hardy space  $H^2(E)$ . This can be treated as a partial converse to the existence-uniqueness theorem (Theorem 3.5 in [10]) of fundamental operators. Therefore, Theorem 4.5 can be rephrased in the following way: every distinguished variety in  $\mathbb{E}$  can be represented as

$$\{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}$$

where  $A_1, A_2$  are the fundamental operators of an  $\mathbb{E}$ -contraction. In Theorem 4.8, we characterize all distinguished varieties for which  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} < 1$ .

We describe Theorem 6.1 again from a different view point; if the adjoint of an  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  is a pure  $\mathbb{E}$ -contraction and if the fundamental operators  $A_1, A_2$  of  $(T_1, T_2, T_3)$  are commuting matrices satisfying  $[A_1^*, A_1] = [A_2^*, A_2]$ , then Theorem 6.1 shows the existence of a one-dimensional subvariety on which von-Neumann's inequality holds. Indeed, Theorem 3.2 provides an  $\mathbb{E}$ -co-isometric extension of such a  $(T_1, T_2, T_3)$  that naturally lives on that subvariety.

In section 5, we describe a connection between the distinguished varieties in the tetrablock with that in the bidisc  $\mathbb{D}^2$  and in the symmetrized bidisc  $\mathbb{G}$ , where

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : |z_1| < 1, |z_2| < 1\} \subseteq \mathbb{C}^2.$$

Indeed, in Theorem 5.5, we show that every distinguished variety in  $\mathbb{E}$  gives rise to a distinguished variety in  $\mathbb{D}^2$  as well as a distinguished variety in  $\mathbb{G}$ .

In section 2, we briefly describe the Taylor joint spectrum of a pair of commuting bounded operators and also recall from the literature some results about the  $\mathbb{E}$ -contractions.

**Note.** After writing this paper, we learned that Theorem 3.4 of this paper has been established independently in [29] by Sau in non-commutative setting.

## 2. TAYLOR JOINT SPECTRUM AND PRELIMINARY RESULTS ABOUT $\mathbb{E}$ -CONTRACTIONS

**2.1. Taylor joint spectrum.** Here we briefly describe the Taylor joint spectrum of a pair of commuting bounded operators and show how in case of commuting matrices it becomes just the set of joint eigenvalues. In fact all notions of joint spectrum (left/right Hart, Taylor) are the same for a pair of commuting matrices. Let  $\underline{T} = (T_1, T_2)$  be a pair commuting bounded operators on a Banach space  $X$ . Let us consider the complex

$$0 \rightarrow X \xrightarrow{\delta_1} X \oplus X \xrightarrow{\delta_2} X \rightarrow 0, \quad (2.1)$$

where  $\delta_1$  and  $\delta_2$  are defined by  $\delta_1 x = T_1 x \oplus T_2 x$  ( $x \in X$ ) and  $\delta_2(x_1 \oplus x_2) = -T_2 x_1 + T_1 x_2$  ( $x_1, x_2 \in X$ ). Clearly  $T_1 T_2 = T_2 T_1$  implies that  $\delta_2 \circ \delta_1 = 0$  so that (2.1) is a chain complex. This chain complex is called the *Koszul complex* of  $\underline{T}$ . To say that the Koszul complex of  $\underline{T}$  is *exact* means three things: at the first and the third stage one has respectively

$$\text{Ker}(T_1) \cap \text{Ker}(T_2) = \{0\} \text{ and } \text{Ran}(\delta_2) = X.$$

In the second stage it means that  $\text{Rand}_1 = \text{Ker}(\delta_2)$ , that is, for every point  $x_1 \oplus x_2 \in X \oplus X$  for which  $-T_2x_1 + T_1x_2 = 0$  has the form  $T_1x \oplus T_2x$  for some  $x \in X$ .  $(T_1, T_2)$  is said to be *Taylor regular* if its Koszul complex is exact. A point  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$  is said to belong to  $\sigma_T(T_1, T_2)$ , the *Taylor joint spectrum* of  $(T_1, T_2)$ , if  $(T_1 - \lambda_1, T_2 - \lambda_2)$  is not Taylor regular. For an explicit description of Taylor joint spectrum in general setting, i.e, of an  $n$ -tuple of commuting bounded operators one can see [31, 32, 25].

We now show that  $\sigma_T(T_1, T_2)$  is just the set of joint eigenvalues of  $(T_1, T_2)$  when  $T_1, T_2$  are matrices. We need the following results before that.

**Lemma 2.1.** *Let  $X_1, X_2$  are Banach spaces and  $A, D$  are bounded operators on  $X_1$  and  $X_2$ . Let  $B \in \mathcal{B}(X_2, X_1)$ . Then*

$$\sigma \left( \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \right) \subseteq \sigma(A) \cup \sigma(D).$$

See Lemma 1 in [22] for a proof of this result.

**Lemma 2.2.** *Let  $\underline{T} = (T_1, T_2)$  be a commuting pair of matrices on an  $N$ -dimensional vector space  $X$ . Then there exists  $N+1$  subspaces  $L_0, L_1, \dots, L_N$  satisfying:*

- (1)  $\{0\} = L_0 \subseteq L_1 \subseteq \dots \subseteq L_N = X$ ,
- (2)  $L_k$  is  $k$ -dimensional for  $k = 1, \dots, N$ ,
- (3) each  $L_k$  is a joint invariant subspace of  $T_1, T_2$ .

*Proof.* It is elementary to see that for a pair of commuting matrices  $(T_1, T_2)$  the set of joint eigenvalues is non-empty. Therefore, there exists a vector  $x_1 \in X$  such that  $x_1$  is a joint eigenvector of  $T_1$  and  $T_2$ . Let  $L_1$  be the one-dimensional subspace spanned by  $x_1$ . Then  $L_1$  is invariant under  $T_1, T_2$ . Next consider the vector space  $Y = X/L_1$  and the linear transformations  $\tilde{T}_1, \tilde{T}_2$  on  $Y$  defined by  $\tilde{T}_i(x+L_1) = T_i x + L_1$ . Then  $\tilde{T}_1, \tilde{T}_2$  are commuting matrices and again they have a joint eigenvalue, say  $(\mu_1, \mu_2)$  and consequently a joint eigenvector, say  $x_2 + L_1$ . Thus  $\tilde{T}_i(x_2 + L_1) = \mu_i x_2 + L_1$  for  $i = 1, 2$  which means that  $T_i x_2 = \mu_i x_2 + z$  for some  $z \in L_1$ . Hence the subspace spanned by  $x_1, x_2$  is invariant under  $T_1, T_2$ . We call this subspace  $L_2$  and it is two-dimensional with  $L_1 \subseteq L_2$ . Now applying the same reasoning to  $X/L_2$  and so on, we get for each  $i = 1, \dots, N-1$  the subspace  $L_i$  spanned by  $x_1, \dots, x_i$ . These subspaces satisfy the conditions of the theorem. Finally, to complete the proof we define  $L_N = X$ . ■

Let us choose an arbitrary  $x_N \in L_N \setminus L_{N-1}$ . Then  $\{x_1, \dots, x_N\}$  is a basis for  $X$  and with respect to this basis the matrices  $T_1, T_2$  are upper-triangular,

i.e., of the form

$$\begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}, \begin{pmatrix} \mu_1 & * & * & * \\ 0 & \mu_2 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_N \end{pmatrix},$$

where each  $(\lambda_i, \mu_i)$  is called a *joint diagonal co-efficient* of  $(T_1, T_2)$ . Let us denote  $\sigma_{dc}(T_1, T_2) = \{(\lambda_i, \mu_i) : i = 1, \dots, N\}$ .

The following result is well known and an interested reader can see [15] for further details. We present here a simple and straight forward proof to this.

**Theorem 2.3.** *Let  $(T_1, T_2)$  be a pair of commuting matrices of order  $N$  and  $\sigma_{pt}(T_1, T_2)$  be the set of joint eigenvalues of  $(T_1, T_2)$ . Then*

$$\sigma_T(T_1, T_2) = \sigma_{pt}(T_1, T_2) = \sigma_{dc}(T_1, T_2).$$

*Proof.* We prove this Theorem by repeated application of Lemma 2.1 to the simultaneous upper-triangularization of Lemma 2.2. It is evident that each  $(\lambda_i, \mu_i)$  is a joint eigenvalue of  $(T_1, T_2)$  and for each  $(\lambda_i, \mu_i)$ ,  $\text{Ker}(T_1 - \lambda_i) \cap \text{Ker}(T_2 - \mu_i) \neq \emptyset$  which means that the Koszul complex (see (2.1)) of  $(T_1 - \lambda_i, T_2 - \mu_i)$  is not exact at the first stage and consequently  $(T_1 - \lambda_i, T_2 - \mu_i)$  is not Taylor-regular. Therefore,  $(\lambda_i, \mu_i) \in \sigma_T(T_1, T_2)$ . Therefore,

$$\sigma_{dc}(T_1, T_2) \subseteq \sigma_{pt}(T_1, T_2) \subseteq \sigma_T(T_1, T_2).$$

Now let  $X_2$  be the subspace spanned by  $x_2, \dots, x_N$ . Then  $X_2$  is  $N - 1$  dimensional and  $X = L_1 \oplus X_2$ . For  $i = 1, 2$  we define  $D_i$  on  $X_2$  by the  $(N - 1) \times (N - 1)$  matrices

$$D_1 = \begin{pmatrix} \lambda_2 & * & * & * \\ 0 & \lambda_3 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}, D_2 = \begin{pmatrix} \mu_2 & * & * & * \\ 0 & \mu_3 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_N \end{pmatrix}.$$

Then  $(D_1, D_2)$  is a commuting pair as  $(T_1, T_2)$  is so. Now we apply Lemma 2.1 and get  $\sigma_T(T_1, T_2) \subseteq \{(\lambda_1, \mu_1)\} \cup \sigma_T(D_1, D_2)$ . Repeating this argument  $N$  times we obtain  $\sigma_T(T_1, T_2) \subseteq \sigma_{dc}(T_1, T_2)$ . Hence we are done.  $\blacksquare$

**2.2. Preliminary results about  $\mathbb{E}$ -contractions.** By virtue of polynomial convexity of  $\mathbb{E}$ , the condition on the Taylor joint spectrum can be avoided and the definition of  $\mathbb{E}$ -contraction can be given only in terms of von-Neumann's inequality as the following lemma shows.

**Lemma 2.4.** *A commuting triple of bounded operators  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction if and only if  $\|f(T_1, T_2, T_3)\| \leq \|f\|_{\infty, \mathbb{E}}$  for any holomorphic polynomial  $f$  in three variables.*



See Lemma 3.3 of [10] for a proof. Let us recall that the *numerical radius* of an operator  $T$  on a Hilbert space  $\mathcal{H}$  is defined by

$$\omega(T) = \sup\{|\langle Tx, x \rangle| : \|x\|_{\mathcal{H}} = 1\}.$$

It is well known that

$$r(T) \leq \omega(T) \leq \|T\| \text{ and } \frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \quad (2.2)$$

where  $r(T)$  is the spectral radius of  $T$ . We state two basic results about numerical radius of which the first result has a routine proof. We shall use these two results in sequel.

**Lemma 2.5.** *The numerical radius of an operator  $T$  is not greater than 1 if and only if  $\operatorname{Re} \beta T \leq I$  for all complex numbers  $\beta$  of modulus 1.*

**Lemma 2.6.** *Let  $A_1, A_2$  be two bounded operators such that  $\omega(A_1 + A_2 z) \leq 1$  for all  $z \in \mathbb{T}$ . Then  $\omega(A_1 + z A_2^*) \leq 1$  and  $\omega(A_1^* + A_2 z) \leq 1$  for all  $z \in \mathbb{T}$ .*

*Proof.* We have that  $\omega(A_1 + z A_2) \leq 1$  for every  $z \in \mathbb{T}$ , which is same as saying that  $\omega(z_1 A_1 + z_2 A_2) \leq 1$  for all complex numbers  $z_1, z_2$  of unit modulus. Thus by Lemma 2.5,

$$(z_1 A_1 + z_2 A_2) + (z_1 A_1 + z_2 A_2)^* \leq 2I,$$

that is

$$(z_1 A_1 + \bar{z}_2 A_2^*) + (z_1 A_1 + \bar{z}_2 A_2^*)^* \leq 2I.$$

Therefore,  $z_1(A_1 + z A_2^*) + \bar{z}_1(A_1 + z A_2^*)^* \leq 2I$  for all  $z, z_1 \in \mathbb{T}$ . This is same as saying that

$$\operatorname{Re} z_1(A_1 + z A_2^*) \leq I, \text{ for all } z, z_1 \in \mathbb{T}.$$

Therefore, by Lemma 2.5 again  $\omega(A_1 + z A_2^*) \leq 1$  for any  $z$  in  $\mathbb{T}$ . The proof of  $\omega(A_1^* + A_2 z) \leq 1$  is similar. ■

We recall from section 1, the existence-uniqueness theorem ([10], Theorem 3.5) for the fundamental operators of an  $\mathbb{E}$ -contraction.

**Theorem 2.7.** *Let  $(T_1, T_2, T_3)$  be an  $\mathbb{E}$ -contraction. Then there are two unique operators  $A_1, A_2$  in  $\mathcal{L}(\mathcal{D}_{T_3})$  such that*

$$T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3} \text{ and } T_2 - T_1^* T_3 = D_{T_3} A_2 D_{T_3}. \quad (2.3)$$

Moreover,  $\omega(A_1 + z A_2) \leq 1$  for all  $z \in \overline{\mathbb{D}}$ .

The following theorem gives a characterization of the set of  $\mathbb{E}$ -unitaries (Theorem 5.4 in [10]).

**Theorem 2.8.** *Let  $\underline{N} = (N_1, N_2, N_3)$  be a commuting triple of bounded operators. Then the following are equivalent.*

- (1)  $\underline{N}$  is an  $\mathbb{E}$ -unitary,
- (2)  $N_3$  is a unitary,  $N_2$  is a contraction and  $N_1 = N_2^* N_3$ ,
- (3)  $N_3$  is a unitary and  $\underline{N}$  is an  $\mathbb{E}$ -contraction.

Here is a structure theorem for the  $\mathbb{E}$ -isometries and a proof can be found in [10] (see Theorem 5.6 and Theorem 5.7 in [10]).

**Theorem 2.9.** *Let  $\underline{V} = (V_1, V_2, V_3)$  be a commuting triple of bounded operators. Then the following are equivalent.*

- (1)  $\underline{V}$  is an  $\mathbb{E}$ -isometry.
- (2)  $\underline{V}$  is an  $\mathbb{E}$ -contraction and  $V_3$  is an isometry.
- (3)  $V_3$  is an isometry,  $V_2$  is a contraction and  $V_1 = V_2^*V_3$ .
- (4) (Wold decomposition)  $\mathcal{H}$  has a decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  into reducing subspaces of  $V_1, V_2, V_3$  such that  $(V_1|_{\mathcal{H}_1}, V_2|_{\mathcal{H}_1}, V_3|_{\mathcal{H}_1})$  is an  $\mathbb{E}$ -unitary and  $(V_1|_{\mathcal{H}_2}, V_2|_{\mathcal{H}_2}, V_3|_{\mathcal{H}_2})$  is a pure  $\mathbb{E}$ -isometry.

### 3. DILATION AND FUNCTIONAL MODEL FOR A SUBCLASS OF PURE $\mathbb{E}$ -CONTRACTIONS

We make a change of notation for an  $\mathbb{E}$ -contraction in this section. Throughout this section we shall denote an  $\mathbb{E}$ -contraction by  $(T_1, T_2, T)$ .

**Proposition 3.1.** *Let  $(Q_1, Q_2, V)$  on  $\mathcal{K}$  be an  $\mathbb{E}$ -isometric dilation of an  $\mathbb{E}$ -contraction  $(T_1, T_2, T)$  on  $\mathcal{H}$ . If  $(Q_1, Q_2, V)$  is minimal, then  $(Q_1^*, Q_2^*, V^*)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1^*, T_2^*, T^*)$ . Conversely, the adjoint of an  $\mathbb{E}$ -co-isometric extension of  $(T_1, T_2, T)$  is an  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, V)$ .*

*Proof.* We first prove that  $T_1P_{\mathcal{H}} = P_{\mathcal{H}}Q_1, T_2P_{\mathcal{H}} = P_{\mathcal{H}}Q_2$  and  $TP_{\mathcal{H}} = P_{\mathcal{H}}V$ . Clearly

$$\mathcal{K} = \overline{\text{span}}\{Q_1^{m_1}Q_2^{m_2}V^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N} \cup \{0\}\}.$$

Now for  $h \in \mathcal{H}$  we have that

$$\begin{aligned} T_1P_{\mathcal{H}}(Q_1^{m_1}Q_2^{m_2}V^n h) &= T_1(T_1^{m_1}T_2^{m_2}T_3^n h) = T_1^{m_1+1}T_2^{m_2}T_3^n h \\ &= P_{\mathcal{H}}(Q_1^{m_1+1}Q_2^{m_2}V^n h) \\ &= P_{\mathcal{H}}Q_1(Q_1^{m_1}Q_2^{m_2}V^n h). \end{aligned}$$

Thus,  $T_1P_{\mathcal{H}} = P_{\mathcal{H}}Q_1$ . Similarly we can prove that  $T_2P_{\mathcal{H}} = P_{\mathcal{H}}Q_2$  and that  $TP_{\mathcal{H}} = P_{\mathcal{H}}V$ . Also for  $h \in \mathcal{H}$  and  $k \in \mathcal{K}$  we have that

$$\begin{aligned} \langle T_1^*h, k \rangle &= \langle P_{\mathcal{H}}T_1^*h, k \rangle = \langle T_1^*h, P_{\mathcal{H}}k \rangle = \langle h, T_1P_{\mathcal{H}}k \rangle = \langle h, P_{\mathcal{H}}Q_1k \rangle \\ &= \langle Q_1^*h, k \rangle. \end{aligned}$$

Hence  $T_1^* = Q_1^*|_{\mathcal{H}}$  and similarly  $T_2^* = Q_2^*|_{\mathcal{H}}$  and  $T^* = V^*|_{\mathcal{H}}$ . Therefore,  $(Q_1^*, Q_2^*, V^*)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1^*, T_2^*, T^*)$ .

The converse part is obvious. ■

Let us recall from [30], the notion of the characteristic function of a contraction  $T$ . For a contraction  $T$  defined on a Hilbert space  $\mathcal{H}$ , let  $\Lambda_T$  be the set of all complex numbers for which the operator  $I - zT^*$  is invertible. For  $z \in \Lambda_T$ , the characteristic function of  $T$  is defined as

$$\Theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|_{\mathcal{D}_T}. \quad (3.1)$$

Here the operators  $D_T$  and  $D_{T^*}$  are the defect operators  $(I - T^*T)^{1/2}$  and  $(I - TT^*)^{1/2}$  respectively. By virtue of the relation  $TD_T = D_{T^*}P$  (section I.3 of [30]),  $\Theta_T(z)$  maps  $\mathcal{D}_T = \overline{\text{Ran}} D_T$  into  $\mathcal{D}_{T^*} = \overline{\text{Ran}} D_{T^*}$  for every  $z$  in  $\Lambda_T$ .

Let us recall that a pure contraction  $T$  is a contraction such that  $T^{*n} \rightarrow 0$  strongly as  $n \rightarrow \infty$ . It was shown in [30] that every pure contraction  $T$  defined on a Hilbert space  $\mathcal{H}$  is unitarily equivalent to the operator  $\mathbb{T} = P_{\mathbb{H}_T}(M_z \otimes I)|_{\mathbb{H}_T}$  on the Hilbert space  $\mathbb{H}_T = (H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}) \ominus M_{\Theta_T}(H^2(\mathbb{D}) \otimes \mathcal{D}_T)$ , where  $M_z$  is the multiplication operator on  $H^2(\mathbb{D})$  and  $M_{\Theta_T}$  is the multiplication operator from  $H^2(\mathbb{D}) \otimes \mathcal{D}_T$  into  $H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}$  corresponding to the multiplier  $\Theta_T$ . Here, in an analogous way, we produce a model for a subclass of pure  $\mathbb{E}$ -contractions  $(T_1, T_2, T)$ .

**Theorem 3.2.** *Let  $(T_1, T_2, T)$  be a pure  $\mathbb{E}$ -contraction on a Hilbert space  $\mathcal{H}$  and let the fundamental operators  $A_{1*}, A_{2*}$  of  $(T_1^*, T_2^*, T^*)$  be commuting operators satisfying  $[A_{1*}^*, A_{1*}] = [A_{2*}^*, A_{2*}]$ . Consider the operators  $Q_1, Q_2, V$  on  $\mathcal{K} = H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}$  defined by*

$$Q_1 = I \otimes A_{1*}^* + M_z \otimes A_{2*}, \quad Q_2 = I \otimes A_{2*}^* + M_z \otimes A_{1*} \text{ and } V = M_z \otimes I.$$

*Then  $(Q_1, Q_2, V)$  is a minimal pure  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, T)$ .*

*Proof.* The minimality is obvious if we prove that  $(Q_1, Q_2, V)$  is an  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, T)$ . This is because  $V$  on  $\mathcal{K}$  is the minimal isometric dilation for  $T$ . Therefore, by virtue of Lemma 3.1, it suffices if we show that  $(Q_1^*, Q_2^*, V^*)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1^*, T_2^*, T^*)$ . Since the fundamental operators  $A_{1*}, A_{2*}$  commute and satisfy  $[A_{1*}^*, A_{1*}] = [A_{2*}^*, A_{2*}]$ ,  $Q_1, Q_2$  and  $V$  commute. Also it is evident that  $V$  is a pure isometry. Thus invoking Theorem 2.9, we need to verify the following steps.

- (1)  $Q_1 = Q_2^*V$ ,  $\|Q_2\| \leq 1$ .
- (2) There is an isometry  $W : \mathcal{H} \rightarrow H^2 \otimes \mathcal{D}_{T^*}$  such that
 
$$Q_1^*|_{W(\mathcal{H})} = WT_1^*W^*|_{W(\mathcal{H})}, \quad Q_2^*|_{W(\mathcal{H})} = WT_2^*W^*|_{W(\mathcal{H})}$$
 and  $V^*|_{W(\mathcal{H})} = WT^*W^*|_{W(\mathcal{H})}$ .

**Step 1.**  $Q_1 = Q_2^*V$  is obvious.  $\|Q_2\| \leq 1$  follows from Lemma 2.6.

**Step 2.** Let us define  $W$  by

$$W : \mathcal{H} \rightarrow \mathcal{K}$$

$$h \mapsto \sum_{n=0}^{\infty} z^n \otimes D_{T^*} T^{*n} h.$$

Now

$$\begin{aligned}
\|Wh\|^2 &= \left\| \sum_{n=0}^{\infty} z^n \otimes D_{T^*} T^{*n} h \right\|^2 \\
&= \left\langle \sum_{n=0}^{\infty} z^n \otimes D_{T^*} T^{*n} h, \sum_{m=0}^{\infty} z^m \otimes D_{T^*} T^{*m} h \right\rangle \\
&= \sum_{m,n=0}^{\infty} \langle z^n, z^m \rangle \langle D_{T^*} T^{*n} h, D_{T^*} T^{*m} h \rangle \\
&= \sum_{n=1}^{\infty} \langle T^n D_{T^*}^2 T^{*n} h, h \rangle \\
&= \sum_{n=0}^{\infty} \langle T^n (I - TT^*) T^{*n} h, h \rangle \\
&= \sum_{n=0}^{\infty} \{ \langle T^n T^{*n} h, h \rangle - \langle T^{n+1} T^{*n+1} h, h \rangle \} \\
&= \|h\|^2 - \lim_{n \rightarrow \infty} \|T^{*n} h\|^2.
\end{aligned}$$

Since  $T$  is a pure contraction,  $\lim_{n \rightarrow \infty} \|T^{*n} h\|^2 = 0$  and hence  $\|Wh\| = \|h\|$ .

Therefore  $W$  is an isometry.

For a basis vector  $z^n \otimes \xi$  of  $\mathcal{K}$  we have that

$$\begin{aligned}
\langle W^*(z^n \otimes \xi), h \rangle &= \langle z^n \otimes \xi, \sum_{k=0}^{\infty} z^k \otimes D_{T^*} T^{*k} h \rangle = \langle \xi, D_{T^*} T^{*n} h \rangle \\
&= \langle T^n D_{T^*} \xi, h \rangle.
\end{aligned}$$

Therefore,

$$W^*(z^n \otimes \xi) = T^n D_{T^*} \xi, \quad \text{for } n = 0, 1, 2, 3, \dots \quad (3.2)$$

and hence

$$TW^*(z^n \otimes \xi) = T^{n+1} D_{T^*} \xi, \quad \text{for } n = 0, 1, 2, 3, \dots$$

Again by (3.2),

$$\begin{aligned}
W^*V(z^n \otimes \xi) &= W^*(M_z \otimes I)(z^n \otimes \xi) = W^*(z^{n+1} \otimes \xi) = T^{n+1} D_{T^*} \xi \\
&= TW^*(z^n \otimes \xi).
\end{aligned}$$

Consequently,  $W^*V = TW^*$ , i.e,  $V^*W = WT^*$  and hence  $V^*|_{W(\mathcal{H})} = WT^*W^*|_{W(\mathcal{H})}$ .

We now show that  $W^*Q_1 = T_1W^*$ .

$$\begin{aligned} W^*Q_1(z^n \otimes \xi) &= W^*(I \otimes A_{1*}^* + M_z \otimes A_{2*})(z^n \otimes \xi) \\ &= W^*(z^n \otimes A_{1*}^*\xi) + W^*(z^{n+1} \otimes A_{2*}\xi) \\ &= T^n D_{T^*} A_{1*}^* \xi + T^{n+1} D_{T^*} A_{2*} \xi. \end{aligned}$$

Also

$$T_1 W^*(z^n \otimes \xi) = T_1 T^n D_{T^*} \xi = T^n T_1 D_{T^*} \xi. \quad (3.3)$$

*Claim.*  $T_1 D_{T^*} = D_{T^*} A_{1*}^* + T D_{T^*} A_{2*}$ .

*Proof of Claim.* Since  $A_{1*}, A_{2*}$  are the fundamental operators of  $(T_1^*, T_2^*, T^*)$ , we have

$$(D_{T^*} A_{1*}^* + T D_{T^*} A_{2*}) D_{T^*} = (T_1 - T T_2^*) + T(T_2^* - T_1 T^*) = T_1 D_{T^*}^2.$$

Now if  $G = T_1 D_{T^*} - D_{T^*} A_{1*}^* - T D_{T^*} A_{2*}$ , then  $G$  is defined from  $\mathcal{D}_{T^*}$  to  $\mathcal{H}$  and  $G D_{T^*} h = 0$  for every  $h \in \mathcal{D}_{T^*}$ . Hence the claim follows.

So from (3.3) we have,

$$T_1 W^*(z^n \otimes \xi) = T^n (D_{T^*} A_{1*}^* + T D_{T^*} A_{2*}).$$

Therefore,  $W^*Q_1 = T_1 W^*$  and hence  $Q_1^*|_{W(\mathcal{H})} = W T_1^* W^*|_{W(\mathcal{H})}$ . Similarly we can show that  $Q_2^*|_{W(\mathcal{H})} = W T_2^* W^*|_{W(\mathcal{H})}$ . The proof is now complete.  $\blacksquare$

In [10], an explicit  $\mathbb{E}$ -isometric dilation was constructed for an  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  whose fundamental operators satisfy  $[A_1, A_2] = 0$  and  $[A_1^*, A_1] = [A_2^*, A_2]$  (Theorem 6.1 in [10]). The fundamental operators of  $(T_1, T_2, T_3)$  were the key ingredients in that construction. Such an explicit  $\mathbb{E}$ -isometric dilation of an  $\mathbb{E}$ -contraction could be treated as an analogue of Schaeffer's construction of isometric dilation of a contraction. The dilation we provided in the previous theorem was only to a pure  $\mathbb{E}$ -contraction and was different in the sense that the fundamental operators of the adjoint  $(T_1^*, T_2^*, T_3^*)$  played the main role there. As a consequence of the dilation theorem in [10], we have the following result.

**Theorem 3.3.** *Let  $T_1, T_2, T_3$  be commuting contractions on a Hilbert space  $\mathcal{H}$ . Let  $A_1, A_2$  be two commuting bounded operators on  $\mathcal{D}_{T_3}$  such that*

$$T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3} \text{ and } T_2 - T_1^* T_3 = D_{T_3} A_2 D_{T_3}.$$

*If  $A_1, A_2$  satisfy  $[A_1^*, A_1] = [A_2^*, A_2]$  and  $\omega(A_1 + A_2 z) \leq 1$ , for all  $z$  from the unit circle, then  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction.*

*Proof.* It is evident from Theorem 6.1 of [10] that such a triple  $(T_1, T_2, T_3)$  has an  $\mathbb{E}$ -isometric dilation and hence an  $\mathbb{E}$ -unitary dilation. Therefore,  $\overline{\mathbb{E}}$  is a complete spectral set for  $(T_1, T_2, T_3)$  and hence  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction.  $\blacksquare$

**Theorem 3.4.** *Let  $(T_1, T_2, T)$  be a pure  $\mathbb{E}$ -contraction on a Hilbert space  $\mathcal{H}$  and let the fundamental operators  $A_{1*}, A_{2*}$  of  $(T_1^*, T_2^*, T^*)$  be commuting operators satisfying  $[A_{1*}^*, A_{1*}] = [A_{2*}^*, A_{2*}]$ . Then  $(T_1, T_2, T)$  is unitarily equivalent to the triple  $(R_1, R_2, R)$  on the Hilbert space  $\mathbb{H}_T = (H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}) \ominus M_{\Theta_T}(H^2(\mathbb{D}) \otimes \mathcal{D}_T)$  defined by*

$$R_1 = P_{\mathbb{H}_T}(I \otimes A_{1*}^* + M_z \otimes A_{2*})|_{\mathbb{H}_T}, \quad R_2 = P_{\mathbb{H}_T}(I \otimes A_{2*}^* + M_z \otimes A_{1*})|_{\mathbb{H}_T}$$

and  $R = P_{\mathbb{H}_T}(M_z \otimes I)|_{\mathbb{H}_T}$ .

*Proof.* It suffices if we show that  $W(\mathcal{H}) = \mathbb{H}_T$ . For this, it is enough if we can prove

$$WW^* + M_{\Theta_T} M_{\Theta_T}^* = I_{H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}}.$$

Since the vectors  $z^n \otimes \xi$  forms a basis for  $H^2(\mathbb{D}) \otimes \mathcal{D}_{T^*}$ , it is obvious from equation (3.3) that

$$W^*(f \otimes \xi) = f(P)D_{P^*}\xi, \text{ for all } f \in \mathbb{C}[z], \text{ and } \xi \in \mathcal{D}_{P^*}.$$

It was shown in the proof of Theorem 1.2 of [8] by Arveson that the operator  $W^*$  satisfies the identity

$$W^*(k_z \otimes \xi) = (I - \bar{z}T)^{-1}D_{T^*}\xi \text{ for } z \in \mathbb{D}, \xi \in \mathcal{D}_{T^*},$$

where  $k_z(w) = (1 - \langle w, z \rangle)^{-1}$ . Therefore, for  $z, w$  in  $\mathbb{D}$  and  $\xi, \eta$  in  $\mathcal{D}_{T^*}$ , we obtain

$$\begin{aligned} & \langle (WW^* + M_{\Theta_T} M_{\Theta_T}^*)k_z \otimes \xi, k_w \otimes \eta \rangle \\ &= \langle W^*(k_z \otimes \xi), W^*(k_w \otimes \eta) \rangle + \langle M_{\Theta_T}^*(k_z \otimes \xi), M_{\Theta_T}^*(k_w \otimes \eta) \rangle \\ &= \langle (I - \bar{z}T)^{-1}D_{T^*}\xi, (I - \bar{w}T)^{-1}D_{T^*}\eta \rangle + \langle k_z \otimes \Theta_T(z)^*\xi, k_w \otimes \Theta_T(w)^*\eta \rangle \\ &= \langle D_{T^*}(I - wT^*)^{-1}(I - \bar{z}T)^{-1}D_{T^*}\xi, \eta \rangle + \langle k_z, k_w \rangle \langle \Theta_T(w)\Theta_T(z)^*\xi, \eta \rangle \\ &= \langle k_z \otimes \xi, k_w \otimes \eta \rangle. \end{aligned}$$

The last equality follows from the following identity (see page 244 in [30]),

$$1 - \Theta_T(w)\Theta_T(z)^* = (1 - w\bar{z})D_{T^*}(1 - wT^*)^{-1}(1 - \bar{z}T)^{-1}D_{T^*},$$

where  $\Theta_T$  is the characteristic function of  $T$ . Using the fact that the vectors  $k_z$  form a total set in  $H^2(\mathbb{D})$ , the assertion follows. ■

**Remark 3.5.** It is interesting to notice that the model space  $\mathbb{H}_T$  and model operator  $R$  are same as the model space and model operator of the pure contraction  $T$  described in [30].

The following theorem, which appeared in [27], gives an explicit model for pure  $\mathbb{E}$ -isometries.

**Theorem 3.6.** *Let  $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$  be a commuting triple of operators on a Hilbert space  $\mathcal{H}$ . If  $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$  is a pure  $\mathbb{E}$ -isometry then there is a unitary operator  $U : \mathcal{H} \rightarrow H^2(\mathcal{D}_{\hat{T}_3}^*)$  such that*

$$\hat{T}_1 = U^*T_\varphi U, \quad \hat{T}_2 = U^*T_\psi U \text{ and } \hat{T}_3 = U^*T_z U,$$

where  $\varphi(z) = A_1^* + A_2z$ ,  $\psi(z) = A_2^* + A_1z$ ,  $z \in \mathbb{D}$  and  $A_1, A_2$  are the fundamental operators of  $(\hat{T}_1^*, \hat{T}_2^*, \hat{T}_3^*)$  such that

- (1)  $[A_1, A_2] = 0$  and  $[A_1^*, A_1] = [A_2^*, A_2]$
- (2)  $\|A_1^* + A_2z\|_{\infty, \mathbb{D}} \leq 1$ .

Conversely, if  $A_1$  and  $A_2$  are two bounded operators on a Hilbert space  $E$  satisfying the above two conditions, then  $(T_{A_1^*+A_2z}, T_{A_2^*+A_1z}, T_z)$  on  $H^2(E)$  is a pure  $\mathbb{E}$ -isometry.

See Theorem 3.3 in [27] for a proof.

**Corollary 3.7.** *Let  $A_1, A_2$  be two commuting operators on a Hilbert space  $E$  which satisfy  $[A_1^*, A_1] = [A_2^*, A_2]$  and  $\omega(A_1 + A_2z) \leq 1$ , for all  $z \in \mathbb{T}$ . Then  $A_1, A_2$  are the fundamental operator of an  $\mathbb{E}$ -contraction on  $H^2(E)$ .*

*Proof.* By Lemma 2.6,  $\omega(A_1^* + A_2z) \leq 1$  and since  $A_1^* + A_2z$  is normal, we have that  $\omega(A_1^* + A_2z) = \|A_1^* + A_2z\| \leq 1$ , for all  $z \in \mathbb{T}$ . Now it is clear from the previous theorem that  $A_1, A_2$  are the fundamental operators of  $(T_{A_1^*+A_2z}^*, T_{A_2^*+A_1z}^*, T_z^*)$  on  $H^2(E)$ .  $\blacksquare$

#### 4. REPRESENTATION OF A DISTINGUISHED VARIETY IN THE TETRABLOCK

We follow here the notations and terminologies used by Agler and M<sup>c</sup>Carthy in [3]. We say that a function  $f$  is *holomorphic* on a distinguished variety  $\Omega$  in  $\mathbb{E}$ , if for every point of  $\Omega$ , there is an open ball  $B$  in  $\mathbb{C}^3$  containing the point and a holomorphic function  $F$  of three variables on  $B$  such that  $F|_{B \cap \Omega} = f|_{B \cap \Omega}$ . We shall denote by  $A(\Omega)$  the Banach algebra of functions that are holomorphic on  $\Omega$  and continuous on  $\overline{\Omega}$ . This is a closed unital subalgebra of  $C(\partial\Omega)$  that separates points. The maximal ideal space of  $A(\Omega)$  is  $\overline{\Omega}$ .

For a finite measure  $\mu$  on  $\Omega$ , let  $H^2(\mu)$  be the closure of polynomials in  $L^2(\partial\Omega, \mu)$ . If  $G$  is an open subset of a Riemann surface  $S$  and  $\nu$  is a finite measure on  $\overline{G}$ , let  $\mathcal{A}^2(\nu)$  denote the closure in  $L^2(\partial G, \nu)$  of  $A(G)$ . A point  $\lambda$  is said to be a *bounded point evaluation* for  $H^2(\mu)$  or  $\mathcal{A}^2(\nu)$  if evaluation at  $\lambda$ , *a priori* defined on a dense set of analytic functions, extends continuously to the whole Hilbert space  $H^2(\mu)$  or  $\mathcal{A}^2(\nu)$  respectively. If  $\lambda$  is a bounded point evaluation, then the function defined by

$$f(\lambda) = \langle f, k_\lambda \rangle$$

is called the *evaluation functional* at  $\lambda$ . The following result is due to Agler and M<sup>c</sup>Carthy (see Lemma 1.1 in [3]).

**Lemma 4.1.** *Let  $S$  be a compact Riemann surface. Let  $G \subseteq S$  be a domain whose boundary is a finite union of piecewise smooth Jordan curves. Then there exists a measure  $\nu$  on  $\partial G$  such that every point  $\lambda$  in  $G$  is a bounded point evaluation for  $\mathcal{A}^2(\nu)$  and such that the linear span of the evaluation functional is dense in  $\mathcal{A}^2(\nu)$ .*

**Lemma 4.2.** *Let  $\Omega$  be a one-dimensional distinguished variety in  $\mathbb{E}$ . Then there exists a measure  $\mu$  on  $\partial\Omega$  such that every point in  $\Omega$  is a bounded point evaluation for  $H^2(\mu)$  and such that the span of the bounded evaluation functionals is dense in  $H^2(\mu)$ .*

*Proof.* Agler and M<sup>c</sup>Carthy proved a similar result for distinguished varieties in the bidisc (see Lemma 1.2 in [3]); we imitate their proof here for the tetrablock.

Let  $p, q$  be minimal polynomials such that

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : p(x_1, x_2, x_3) = q(x_1, x_2, x_3) = 0\}.$$

Let  $\mathbb{Z}_{pq}$  be the intersection of the zero sets of  $p$  and  $q$ , i.e.,  $\mathbb{Z}_{pq} = \mathbb{Z}_p \cap \mathbb{Z}_q$ . Let  $C(\mathbb{Z}_{pq})$  be the closure of  $\mathbb{Z}_{pq}$  in the projective space  $\mathbb{CP}^3$ . Let  $S$  be the desingularization of  $C(\mathbb{Z}_{pq})$ . See, e.g., [18], [20] and [21] for details of desingularization. Therefore,  $S$  is a compact Riemann surface and there is a holomorphic map  $\tau : S \rightarrow C(\mathbb{Z}_{pq})$  that is biholomorphic from  $S'$  onto  $C(\mathbb{Z}_{pq})'$  and finite-to-one from  $S \setminus S'$  onto  $C(\mathbb{Z}_{pq}) \setminus C(\mathbb{Z}_{pq})'$ . Here  $C(\mathbb{Z}_{pq})'$  is the set of non-singular points in  $C(\mathbb{Z}_{pq})$  and  $S'$  is the pre-image of  $C(\mathbb{Z}_{pq})'$  under  $\tau$ .

Let  $G = \tau^{-1}(\Omega)$ . Then  $\partial G$  is a finite union of disjoint curves, each of which is analytic except possibly at a finite number of cusps and  $G$  satisfies the conditions of Lemma 4.1. So there exists a measure  $\nu$  on  $\partial G$  such that every point in  $G$  is a bounded point evaluation for  $\mathcal{A}^2(\nu)$ . Let us define our desired measure  $\mu$  by

$$\mu(E) = \nu(\tau^{-1}(E)), \text{ for a Borel subset } E \text{ of } \partial\Omega.$$

Clearly, if  $\lambda$  is in  $G$  and  $\tau(\eta) = \lambda$ , let  $k_\eta \nu$  be a representing measure for  $\eta$  in  $A(G)$ . Then the function  $k_\eta \circ \tau^{-1}$  is defined  $\mu$ -almost everywhere and satisfies

$$\begin{aligned} \int_{\partial\Omega} p(k_\eta \circ \tau^{-1}) d\mu &= \int_{\partial G} (p \circ \tau) k_\eta d\nu = p \circ \tau(\eta) = p(\lambda) \text{ and} \\ \int_{\partial\Omega} q(k_\eta \circ \tau^{-1}) d\mu &= \int_{\partial G} (q \circ \tau) k_\eta d\nu = q \circ \tau(\eta) = q(\lambda). \end{aligned}$$

■

**Lemma 4.3.** *Let  $\Omega$  be a one-dimensional distinguished variety in  $\mathbb{E}$ , and let  $\mu$  be the measure on  $\partial\Omega$  given as in Lemma 4.2. A point  $(y_1, y_2, y_3) \in \mathbb{E}$  is in  $\Omega$  if and only if  $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$  is a joint eigenvalue for  $M_{x_1}^*, M_{x_2}^*$  and  $M_{x_3}^*$ .*

*Proof.* It is a well known fact in the theory of reproducing kernel Hilbert spaces that  $M_f^* k_x = \overline{f(x)} k_x$  for every multiplier  $f$  and every kernel function  $k_x$ ; in particular every point  $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \Omega$  is a joint eigenvalue for  $M_{y_1}^*, M_{y_2}^*$  and  $M_{y_3}^*$ .

Conversely, if  $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$  is a joint eigenvalue and  $v$  is a unit eigenvector, then  $f(y_1, y_2, y_3) = \langle v, M_f^* v \rangle$  for every polynomial  $f$ . Therefore,

$$|f(y_1, y_2, y_3)| \leq \|M_f\| = \sup_{(x_1, x_2, x_3) \in \Omega} |f(x_1, x_2, x_3)|.$$



So  $(y_1, y_2, y_3)$  is in the polynomial convex hull of  $\Omega$  (relative to  $\mathbb{E}$ ), which is  $\Omega$ .  $\blacksquare$

**Lemma 4.4.** *Let  $\Omega$  be a one-dimensional distinguished variety in  $\mathbb{E}$ , and let  $\mu$  be the measure on  $\partial\Omega$  given as in Lemma 4.2. The multiplication operator triple  $(M_{x_1}, M_{x_2}, M_{x_3})$  on  $H^2(\mu)$ , defined as multiplication by the co-ordinate functions, is a pure  $\mathbb{E}$ -isometry.*

*Proof.* Let us consider the pair of operators  $(\widehat{M_{x_1}}, \widehat{M_{x_2}}, \widehat{M_{x_3}})$ , multiplication by co-ordinate functions, on  $L^2(\partial\Omega, \mu)$ . They are commuting normal operators and the joint spectrum  $\mu(\widehat{M_{x_1}}, \widehat{M_{x_2}}, \widehat{M_{x_3}})$  is contained in  $\partial\Omega \subseteq b\mathbb{E}$ . Therefore,  $(\widehat{M_{x_1}}, \widehat{M_{x_2}}, \widehat{M_{x_3}})$  is an  $\mathbb{E}$ -unitary and  $(M_{x_1}, M_{x_2}, M_{x_3})$ , being the restriction of  $(\widehat{M_{x_1}}, \widehat{M_{x_2}}, \widehat{M_{x_3}})$  to the common invariant subspace  $H^2(\mu)$ , is an  $\mathbb{E}$ -isometry. By a standard computation, for every  $\bar{y} = (y_1, y_2, y_3) \in \Omega$ , the kernel function  $k_{\bar{y}}$  is an eigenfunction of  $M_{x_3}^*$  corresponding to the eigenvalue  $\bar{y}_3$ . Therefore,

$$(M_{x_3}^*)^n k_{\bar{y}} = \bar{y}_3^n k_{\bar{y}} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

because  $|y_3| < 1$  by Theorem 1.1. Since the evaluation functionals  $k_{\bar{y}}$  are dense in  $H^2(\mu)$ , this shows that  $M_{x_3}$  is pure. Hence  $M_{x_3}$  is a pure isometry and consequently  $(M_{x_1}, M_{x_2}, M_{x_3})$  is a pure  $\mathbb{E}$ -isometry on  $H^2(\mu)$ .  $\blacksquare$

We now present the main result of this section, the theorem that gives a representation of a distinguished variety in  $\mathbb{E}$  in terms of the natural coordinates of  $\mathbb{E}$ .

**Theorem 4.5.** *Let*

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}, \quad (4.1)$$

where  $A_1, A_2$  are commuting square matrices of same order such that

- (1)  $[A_1^*, A_1] = [A_2^*, A_2]$
- (2)  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} < 1$ .

Then  $\Omega$  is a one-dimensional distinguished variety in  $\mathbb{E}$ . Conversely, every distinguished variety in  $\mathbb{E}$  is one-dimensional and can be represented as (4.1) for two commuting square matrices  $A_1, A_2$  of same order, such that

- (1)  $[A_1^*, A_1] = [A_2^*, A_2]$
- (2)  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} \leq 1$ .

*Proof.* Suppose that

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\},$$

where  $A_1, A_2$  are commuting matrices of order  $n$  satisfying the given conditions. Then for any  $x_3$ ,  $A_1^* + x_3 A_2$  and  $A_2^* + x_3 A_1$  commute and consequently  $\sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1) \neq \emptyset$ . We now show that if  $|x_3| < 1$  and  $(x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)$  then  $(x_1, x_2, x_3) \in \Omega$  which will establish that  $\Omega$  is non-empty and that it exits through the distinguished boundary  $b\mathbb{E}$ . This is because proving the fact that  $\Omega$  exits through  $b\mathbb{E}$  is

same as proving that  $\overline{\Omega} \cap (\partial E \setminus bE) = \emptyset$ , i.e, if  $(x_1, x_2, x_3) \in \overline{\Omega}$  and  $|x_3| < 1$  then  $(x_1, x_2, x_3) \in \mathbb{E}$  (by Theorem 1.1). Let  $|x_3| < 1$  and  $(x_1, x_2)$  be a joint eigenvalue of  $A_1^* + x_3 A_2$  and  $A_2^* + x_3 A_1$ . Then there exists a unit joint eigenvector  $\nu$  such that  $(A_1^* + x_3 A_2)\nu = x_1 \nu$  and  $(A_2^* + x_3 A_1)\nu = x_2 \nu$ . Taking inner product with respect to  $\nu$  we get

$$\alpha_1 + \bar{\beta}_1 x_3 = x_1 \text{ and } \beta_1 + \bar{\alpha}_1 x_3 = x_2,$$

where  $\alpha_1 = \langle A_1^* \nu, \nu \rangle$  and  $\beta_1 = \langle A_2^* \nu, \nu \rangle$ . Here  $\alpha_1$  and  $\beta_1$  are unique because we have that  $x_1 - \bar{x}_2 x_3 = \alpha_1(1 - |x_3|^2)$  and  $x_2 - \bar{x}_1 x_3 = \beta_1(1 - |x_3|^2)$  which lead to

$$\alpha_1 = \frac{x_1 - \bar{x}_2 x_3}{1 - |x_3|^2} \text{ and } \beta_1 = \frac{x_2 - \bar{x}_1 x_3}{1 - |x_3|^2}.$$

Since  $A_1, A_2$  commute and  $[A_1^*, A_1] = [A_2^*, A_2]$ ,  $A_1^* + A_2 z$  is a normal matrix for every  $z$  of unit modulus. So we have that  $\|A_1^* + A_2 z\| = \omega(A_1^* + A_2 z) < 1$  and by Lemma 2.6,  $\omega(A_1 + A_2 z) < 1$  for every  $z$  in  $\mathbb{T}$ . This implies that  $\omega(z_1 A_1^* + z_2 A_2^*) < 1$ , for every  $z_1, z_2$  in  $\mathbb{T}$  and hence

$$|z_1 \langle A_1^* \nu, \nu \rangle + z_2 \langle A_2^* \nu, \nu \rangle| < 1, \text{ for every } z_1, z_2 \in \mathbb{T}.$$

If both  $\langle A_1^* \nu, \nu \rangle$  and  $\langle A_2^* \nu, \nu \rangle$  are non-zero, we can choose  $z_1 = \frac{|\langle A_1^* \nu, \nu \rangle|}{\langle A_1^* \nu, \nu \rangle}$  and  $z_2 = \frac{|\langle A_2^* \nu, \nu \rangle|}{\langle A_2^* \nu, \nu \rangle}$  to get  $|\alpha_1| + |\beta_1| < 1$ . If any of them or both  $\langle A_1^* \nu, \nu \rangle$  and  $\langle A_2^* \nu, \nu \rangle$  are zero then also  $|\alpha_1| + |\beta_1| < 1$ . Therefore, by Theorem 1.1,  $(x_1, x_2, x_3)$  is in  $\mathbb{E}$ . Thus,  $\Omega$  is non-empty and it exits through the distinguished boundary  $b\mathbb{E}$ .

Again for any  $x_3$ , there is a unitary matrix  $U$  of order  $n$  (see Lemma 2.2) such that  $U^*(A_1^* + x_3 A_2)U$  and  $U^*(A_2^* + x_3 A_1)U$  have the following upper triangular form

$$U^*(A_1^* + x_3 A_2)U = \begin{pmatrix} \alpha_1 + \bar{\beta}_1 x_3 & * & * & * \\ 0 & \alpha_2 + \bar{\beta}_2 x_3 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n + \bar{\beta}_n x_3 \end{pmatrix},$$

$$U^*(A_2^* + x_3 A_1)U = \begin{pmatrix} \beta_1 + \bar{\alpha}_1 x_3 & * & * & * \\ 0 & \beta_2 + \bar{\alpha}_2 x_3 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n + \bar{\alpha}_n x_3 \end{pmatrix}$$

and the joint spectrum  $\sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)$  can be read off the diagonal of the common triangular form. It is evident from definition that  $\Omega$  has dimension one. Thus it remains to show that  $\Omega$  is a variety in  $\mathbb{E}$ . We show that  $\Omega$  is a variety in  $\mathbb{E}$  determined by the ideal generated by the set of polynomials

$$\mathcal{F} = \{\det[z_1(A_1^* + x_3 A_2 - x_1 I) + z_2(A_2^* + x_3 A_1 - x_2 I)] = 0 : z_1, z_2 \in \overline{\mathbb{D}}\}.$$

This is same as showing that  $\mathbb{E} \cap \mathbb{Z}(\mathcal{F}) = \Omega$ ,  $\mathbb{Z}(\mathcal{F})$  being the variety determined by the ideal generated by  $\mathcal{F}$ . Let  $(x_1, x_2, x_3) \in \Omega$ . Then  $x_1 = \alpha_k + \bar{\beta}_k x_3$  and  $x_2 = \beta_k + \bar{\alpha}_k x_3$  for some  $k$  between 1 and  $n$ . Therefore,  $z_1(\alpha_k + \bar{\beta}_k x_3 - x_1) + z_2(\beta_k + \bar{\alpha}_k x_3 - x_2) = 0$  for any  $z_1, z_2$  in  $\overline{\mathbb{D}}$  and consequently  $(x_1, x_2, x_3) \in \mathbb{Z}(\mathcal{F}) \cap \mathbb{E}$ . Again let  $(x_1, x_2, x_3) \in \mathbb{Z}(\mathcal{F}) \cap \mathbb{E}$ . Then  $\det[z_1(A_1^* + x_3 A_2 - x_1 I) + z_2(A_2^* + x_3 A_1 - x_2 I)] = 0$  for all  $z_1, z_2 \in \overline{\mathbb{D}}$  which implies that the two matrices  $A_1^* + x_3 A_2 - x_1 I$  and  $A_2^* + x_3 A_1 - x_2 I$  have 0 at a common position in their diagonals. Thus  $(x_1, x_2)$  is a joint eigenvalue of  $A_1^* + x_3 A_2$  and  $A_2^* + x_3 A_1$  and  $(x_1, x_2, x_3) \in \Omega$ . Hence  $\Omega = \mathbb{Z}(\mathcal{F}) \cap \mathbb{E}$  and  $\Omega$  is a distinguished variety in  $\mathbb{E}$ . The ideal generated by  $\mathcal{F}$  must have a finite set of generators and we leave it to an interested reader to determine such a finite set.

Conversely, let  $\Omega$  be a distinguished variety in  $\mathbb{E}$ . We first show that  $\Omega$  cannot be a two-dimensional complex algebraic variety. Let if possible  $\Omega$  be two-dimensional and determined by a single polynomial  $p$  in three variables, i.e.,

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : p(x_1, x_2, x_3) = 0\}.$$

Let  $(y_1, y_2, y_3) \in \Omega$ . Therefore,  $|y_3| < 1$ . We show that  $\overline{\Omega}$  has intersection with  $\partial\mathbb{E} \setminus b\mathbb{E}$  which proves that  $\Omega$  does not exit through the distinguished boundary. Let  $S_{y_3}$  be the set of all points in  $\Omega$  with  $y_3$  as the third coordinate, i.e.,  $S_{y_3} = \{(x_1, x_2, x_3) \in \Omega : x_3 = y_3\}$ . Such  $(x_1, x_2)$  are the zeros of the polynomial  $p(x_1, x_2, y_3)$ . If  $x_1 = x_2$  for every  $(x_1, x_2, y_3) \in S_{y_3}$ , then  $p(x_1, x_2, y_3)$  becomes a polynomial in one variable and consequently  $S_{y_3}$  is a finite set. If every such  $S_{y_3}$  is a finite set then  $\Omega$  becomes a one dimensional variety, a contradiction. Therefore, there exists  $y_3$  such that  $p(x_1, x_2, y_3)$  gives a one-dimensional variety and consequently  $S_{y_3}$  is not a finite set. We choose such  $y_3$ . Since each  $(x_1, x_2, y_3)$  in  $S_{y_3}$  is a point in  $\mathbb{E}$ , by Theorem 1.1, there exist complex numbers  $\beta_1, \beta_2$  with  $|\beta_1| + |\beta_2| < 1$  such that  $x_1 = \beta_1 + \bar{\beta}_2 y_3$  and  $x_2 = \beta_2 + \bar{\beta}_1 y_3$ . Let us consider the domain  $G$  defined by

$$G = \{(\beta_1, \beta_2) \in \mathbb{C}^2 : |\beta_1| + |\beta_2| < 1\},$$

and the map

$$\begin{aligned} \varpi : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (\beta_1, \beta_2) &\mapsto (\beta_1 + \bar{\beta}_2 y_3, \beta_2 + \bar{\beta}_1 y_3). \end{aligned}$$

It is evident that the points  $(x_1, x_2)$  for which  $(x_1, x_2, y_3) \in S_{y_3}$  lie inside  $\varpi(G)$ . Also it is clear that  $\varpi$  maps  $G$  into  $\mathbb{D}^2$  because the tetrablock lives inside  $\mathbb{D}^3$ . This map  $\varpi$  is real-linear and invertible when considered a map from  $\mathbb{R}^4$  to  $\mathbb{R}^4$ , in fact a homeomorphism of  $\mathbb{R}^4$ . Therefore,  $\varpi$  is open and it maps the boundary of  $G$  onto the boundary of  $\varpi(G)$ . Therefore, the zero-set of the polynomial  $p(x_1, x_2, y_3)$  in two variables ( $y_3$  being a constant) is a one-dimensional variety a part of which lies inside  $\varpi(G)$ . Therefore, this variety intersects the boundary of the domain  $\varpi(G)$  which is the image of

the set  $\{(\beta_1, \beta_2) \in \mathbb{C}^2 : |\beta_1| + |\beta_2| = 1\}$  under  $\varpi$ . Thus, there is a point  $(\lambda_1, \lambda_2, y_3)$  in the zero set of  $p$  such that  $\lambda_1 = \beta_1 + \bar{\beta}_2 y_3$  and  $\lambda_2 = \beta_2 + \bar{\beta}_1 y_3$  with  $|\beta_1| + |\beta_2| = 1$ . Therefore,  $(\lambda_1, \lambda_2, y_3) \in \bar{\Omega} \cap \bar{\mathbb{E}}$ . Since  $|y_3| < 1$ ,  $(\lambda_1, \lambda_2, y_3) \in \partial \mathbb{E} \setminus b\mathbb{E}$  and consequently  $\Omega$  is not a distinguished variety, a contradiction. Thus, there is no two-dimensional distinguished variety in  $\mathbb{E}$  and  $\Omega$  is one-dimensional.

Let  $p_1, \dots, p_n$ , ( $n > 1$ ) be polynomials in three variables such that

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : p_1(x_1, x_2, x_3) = \dots = p_n(x_1, x_2, x_3) = 0\}.$$

We claim that all  $p_i$  cannot be divisible by  $x_3$ . Indeed, if  $p_i$  is divisible by  $x_3$  for all  $i$  then  $p_1 = \dots = p_n = 0$  when  $x_3 = 0$ . The point  $(0, 1, 0) \in \bar{\mathbb{E}}$  (by choosing  $\beta_1 = 0, \beta_2 = 1$  and applying Theorem 1.1) and clearly  $p_i(0, 1, 0) = 0$  for each  $i$  but  $(0, 1, 0) \notin b\mathbb{E}$  although  $(0, 1, 0) \in \partial \mathbb{E}$  as  $|\beta_1| + |\beta_2| = 1$ . This leads to the conclusion that  $\Omega$  does not exit through the distinguished boundary of  $\mathbb{E}$ , a contradiction. Therefore, all  $p_1, \dots, p_n$  are not divisible by  $x_3$ . Let  $p_1$  be not divisible by  $x_3$  and

$$p_1(x_1, x_2, x_3) = \sum_{\substack{0 \leq i \leq m_1 \\ 0 \leq j \leq m_2}} a_{ij} x_1^i x_2^j + x_3 r(x_1, x_2, x_3), \quad (4.2)$$

for some polynomial  $r$  and  $a_{m_1 m_2} \neq 0$ .

Let  $(M_{x_1}, M_{x_2}, M_{x_3})$  be the triple of operators on  $H^2(\mu)$  given by the multiplication by co-ordinate functions, where  $\mu$  is the measure as in Lemma 4.2. Then by Lemma 4.4,  $(M_{x_1}, M_{x_2}, M_{x_3})$  is a pure  $\mathbb{E}$ -isometry on  $H^2(\mu)$ . Now  $M_{x_3} M_{x_3}^*$  is a projection onto  $\text{Ran } M_{x_3}$  and

$$\text{Ran } M_{x_3} \supseteq \{x_3 f(x_1, x_2, x_3) : f \text{ is a polynomial in } x_1, x_2, x_3\}.$$

It is evident from (4.2) that

$$a_{lk} x_1^l x_2^k \in \overline{\text{Ran } M_{x_3}} \oplus \overline{\text{span}}\{x_1^i x_2^j : 1 \leq i \leq m_1, 1 \leq j \leq m_2, i \neq l, j \neq k\},$$

for each  $k, l$  and hence

$$H^2(\mu) = \overline{\text{Ran } M_{x_3}} \oplus \overline{\text{span}}\{x_1^i x_2^j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\}.$$

Therefore,  $\text{Ran } (I - M_{x_3} M_{x_3}^*)$  has finite dimension, say  $n$ . By Theorem 3.6,  $(M_{x_1}, M_{x_2}, M_{x_3})$  can be identified with  $(T_\varphi, T_\psi, T_z)$  on  $H^2(\mathcal{D}_{M_{T_3}^*})$ , where  $\varphi(z) = A_1^* + A_2 z$  and  $\psi(z) = A_2^* + A_1 z$ ,  $A_1, A_2$  being the fundamental operators of  $(T_{A_1^* + A_2 z}^*, T_{A_2^* + A_1 z}^*, T_z^*)$ . By Lemma 4.3, a point  $(y_1, y_2, y_3)$  is in  $\Omega$  if and only if  $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$  is a joint eigenvalue of  $T_\varphi^*, T_\psi^*$  and  $T_z^*$ . This can happen if and only if  $(\bar{y}_1, \bar{y}_2)$  is a joint eigenvalue of  $\varphi(y_3)^*$  and  $\psi(y_3)^*$ . This leads to

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}.$$

By the commutativity of  $T_\varphi$  and  $T_\psi$  we have that  $[A_1, A_2] = 0$  and that  $[A_1^*, A_1] = [A_2^*, A_2]$ . The proof is now complete. ■

A variety given by the determinantal representation (4.1), where  $A_1, A_2$  satisfy  $\|A_1^* + A_2 z\|_{\infty, \mathbb{T}} = 1$ , may or may not be a distinguished variety in the tetrablock as the following examples illustrate.

**Example 4.6.** Let us consider the commuting self-adjoint matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then for any  $z$  of unit modulus

$$A + Bz = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is a normal matrix and for a unit vector

$$h = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \in \mathbb{C}^3,$$

we have that  $\|(A + Bz)h\| = \|h\|$  and therefore,  $\|A + Bz\| = \omega(A + Bz) = 1$ . Now we define

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A + x_3 B, B + x_3 A)\}.$$

Clearly  $(1, 0, 0) \in \partial\mathbb{E} \cap \overline{\Omega}$ , by Theorem 1.1 (by choosing  $x_3 = 0, \beta_1 = 1$  and  $\beta_2 = 0$ ) but  $(1, 0, 0) \notin b\mathbb{E}$  which shows that  $\Omega$  does not exit through the distinguished boundary  $b\mathbb{E}$ . Hence  $\Omega$  is not a distinguished variety.

**Example 4.7.** Let

$$A_1 = A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A_1^* + A_2 z = A_1^* + A_1 z = \begin{pmatrix} 0 & z & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and  $\|A_1^* + A_1 z\| = 1$ , for all  $z \in \mathbb{T}$ . Let

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_1, A_1^* + x_3 A_1)\}$$

Here

$$A_1^* + x_3 A_1 - xI = \begin{pmatrix} -x & x_3 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{pmatrix}$$

and thus

$$\det(A_1^* + x_3 A_1 - xI) = x(x_3 - x^2).$$

Clearly  $\Omega$  is non-empty as it contains the points of the form  $(0, 0, x_3)$ . Clearly this sheet of the variety  $\Omega$  exits through  $b\mathbb{E}$ . It is evident that  $\Omega$  is one-dimensional. Also,  $\Omega$  contains the points  $(x, x, x_3)$  with  $x^2 = x_3$ . By Theorem 1.1, we have that  $x = \beta_1 + \beta_2 x_3 = \beta_2 + \beta_1 x_3$ , for some  $\beta_1, \beta_2$  with  $|\beta_1| + |\beta_2| \leq 1$ . Now when  $x_3 \neq 0$ ,  $x \neq 0$  and hence  $(\beta_1, \beta_2) \neq (0, 0)$ . When  $\beta_1 \neq \beta_2$ , we have

$$|x_3| = \left| \frac{\beta_1 - \beta_2}{\bar{\beta}_1 - \bar{\beta}_2} \right| = 1$$

and hence  $(x, x, x_3) \in b\mathbb{E}$ . When  $\beta_1 = \beta_2 = \beta$ , we show that  $(x, x, x_3) \in \mathbb{E}$  if  $|x_3| < 1$ . Let  $|x_3| < 1$  and  $x = \beta + \bar{\beta}x_3$ . It suffices to show that  $|\beta| + |\bar{\beta}| < 1$ , i.e,  $|\beta| < 1/2$ . Let if possible  $|\beta| = 1/2$  and  $\beta = \frac{1}{2}e^{i\theta}$ . Since  $x^2 = x_3$ , without loss of generality let  $x = \sqrt{x_3}$ . So we have

$$\sqrt{x_3} = \beta + \bar{\beta}x_3 = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}x_3,$$

which implies that  $(\sqrt{x_3} - e^{i\theta})^2 = 0$ . Therefore,  $|x_3| = 1$ , a contradiction. Thus  $|\beta| < 1/2$  and  $(x, x, x_3) \in \mathbb{E}$ . Therefore,  $\bar{\Omega} \cap \partial\mathbb{E} = \bar{\Omega} \cap b\mathbb{E}$  and  $\Omega$  is a distinguished variety.

Below we characterize all distinguished varieties for which  $\|A_1 + A_2 z\|_{\infty, \mathbb{T}} < 1$ .

**Theorem 4.8.** *Let  $\Omega$  be a variety in  $\mathbb{E}$ . Then*

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}$$

*for two commuting square matrices  $A_1, A_2$  satisfying  $[A_1^*, A_1] = [A_2^*, A_2]$  and  $\|A_1 + A_2 z\|_{\infty, \mathbb{T}} < 1$  if and only if  $\Omega$  is a distinguished variety in  $\mathbb{E}$  such that  $\partial\Omega \cap bD_{\mathbb{E}} = \emptyset$ , where*

$$bD_{\mathbb{E}} = \{(x_1, x_2, x_1 x_2) : |x_1| = |x_2| = 1\}.$$

*Proof.* Recall that

$$\begin{aligned} b\mathbb{E} &= \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 = \bar{x}_2 x_3, |x_3| = 1 \text{ and } |x_2| \leq 1\} \\ &= \{(x_1, x_2, x_3) \in \bar{\mathbb{E}} : |x_3| = 1\}. \end{aligned}$$

It is clear that  $bD_{\mathbb{E}} \subseteq b\mathbb{E}$ . If  $\Omega$  has such an expression in terms of joint eigenvalues of  $A_1^* + A_2 z$  and  $A_2^* + A_1 z$  then by Theorem 4.5,  $\Omega$  is a distinguished variety in  $\mathbb{E}$ . We show that  $\partial\Omega \cap bD_{\mathbb{E}} = \emptyset$ . Obviously  $A_1^* + A_2 z$  and  $A_2^* + A_1 z$  are commuting normal matrices for every  $z \in \mathbb{T}$ . If  $(x_1, x_2, e^{i\theta}) \in \partial\Omega$ , then  $(x_1, x_2)$  is a joint eigenvalue of  $A_1^* + e^{i\theta} A_2$  and  $A_2^* + e^{i\theta} A_1$ . But  $\|A_1^* + A_2 z\| = \omega(A_1^* + A_2 z) < 1$  and hence  $|x_i| < 1$  for  $i = 1, 2$ . Therefore,  $(x_1, x_2, e^{i\theta}) \notin bD_{\mathbb{E}}$ .

Conversely, suppose that  $\Omega$  is a distinguished variety such that  $\partial\Omega \cap bD_{\mathbb{E}} = \emptyset$ . In course of the proof of Theorem 4.5 we showed that  $\Omega$  is given by (4.1) with  $A_1, A_2$  being the fundamental operators of  $(M_{x_1}^*, M_{x_2}^*, M_{x_3}^*)$  on  $H^2(\mu)$ . What we need to show is that  $\|A_1 + A_2 z\|_{\infty, \mathbb{T}} < 1$ .

We saw in the proof of the Theorem 4.5 that  $(M_{x_1}, M_{x_2}, M_{x_3})$  is unitarily equivalent to  $(T_\varphi, T_\psi, T_z)$  on  $H^2(\mathcal{D}_{M_{T_3}^*})$ , where  $\varphi(z) = A_1^* + A_2 z$  and  $\psi(z) = A_2^* + A_1 z$ , for two commuting matrices  $A_1, A_2$  satisfying  $[A_1^*, A_1] = [A_2^*, A_2]$ . Since  $\partial\Omega \cap bD_{\mathbb{E}} = \emptyset$ , we have that  $\|M_{x_1}^*\| < 1$  and  $\|M_{x_2}^*\| < 1$ . Therefore,

$$\|T_\varphi\| = \|A_1^* + A_2 z\|_{\infty, \mathbb{T}} = \sup_{z \in \mathbb{T}} \omega(A_1^* + A_2 z) < 1$$

and the proof is complete. ■

## 5. A CONNECTION WITH THE BIDISC AND THE SYMMETRIZED BIDISC

Recall that the symmetrized bidisc  $\mathbb{G}$ , its closure  $\Gamma$  and the distinguished boundary  $b\Gamma$  are the following sets:

$$\begin{aligned} \mathbb{G} &= \{(z_1 + z_2, z_1 z_2) : |z_1| < 1, |z_2| < 1\} \subseteq \mathbb{C}^2; \\ \Gamma &= \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1, |z_2| \leq 1\}; \\ b\Gamma &= \{(z_1 + z_2, z_1 z_2) : |z_1| = 1, |z_2| = 1\} \\ &= \{(s, p) \in \Gamma : |p| = 1\}. \end{aligned}$$

A pair of commuting operators  $(S, P)$  on a Hilbert space  $\mathcal{H}$  that has  $\Gamma$  as a spectral set, is called a  $\Gamma$ -contraction. The symmetrized bidisc enjoys rich operator theory [5, 6, 11, 12, 26]. Operator theory and complex geometry of the tetrablock have beautiful connections with that of the symmetrized bidisc as was shown in [10]. We state here two of the important results in this line from [10].

**Lemma 5.1.** *A point  $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$  if and only if  $(x_1 + zx_2, zx_3) \in \Gamma$  for every  $z$  on the unit circle.*

**Theorem 5.2.** *Let  $(T_1, T_2, T_3)$  be an  $\mathbb{E}$ -contraction. Then  $(T_1 + zT_2, zT_3)$  is a  $\Gamma$ -contraction for every  $z$  of unit modulus.*

See Lemma 3.2 and Theorem 3.5 respectively in [10] for details. The distinguished varieties in the symmetrized bidisc, their representations and relations with the operator theory have been described beautifully in [28]. We recall from [28] Lemma 3.1 and Theorem 3.5 which will help proving the main result of this section, Theorem 5.5.

**Lemma 5.3.** *Let  $W \subseteq \mathbb{G}$ . Then  $W$  is a distinguished variety in  $\mathbb{G}$  if and only if there is a distinguished variety  $V$  in  $\mathbb{D}^2$  such that  $W = \pi(V)$ , where  $\pi$  is the symmetrization map from  $\mathbb{C}^2$  to  $\mathbb{C}^2$  that maps  $(z_1, z_2)$  to  $(z_1 + z_2, z_1 z_2)$ .*

**Theorem 5.4.** *Let  $A$  be a square matrix  $A$  with  $\omega(A) < 1$ , and let  $W$  be the subset of  $\mathbb{G}$  defined by*

$$W = \{(s, p) \in \mathbb{G} : \det(A + pA^* - sI) = 0\}.$$

*Then  $W$  is a distinguished variety. Conversely, every distinguished variety in  $\mathbb{G}$  has the form  $\{(s, p) \in \mathbb{G} : \det(A + pA^* - sI) = 0\}$ , for some matrix  $A$  with  $\omega(A) \leq 1$ .*

Let us consider the holomorphic map whose source is Lemma 5.1:

$$\begin{aligned}\phi : \overline{\mathbb{E}} &\longrightarrow \Gamma \\ (x_1, x_2, x_3) &\mapsto (x_1 + x_2, x_3).\end{aligned}$$

**Theorem 5.5.** *If*

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{E} : (x_1, x_2) \in \sigma_T(A_1^* + A_2x_3, A_2^* + A_1x_3)\}$$

*is a distinguished variety in  $\mathbb{E}$ , then  $W = \phi(\Omega)$  is a distinguished variety in  $\mathbb{G}$  provided that  $\|A_1^* + A_2z\|_{\infty, \mathbb{T}} < 1$ . Moreover,  $\Omega$  gives rise to a distinguished variety in  $\mathbb{D}^2$ .*

*Proof.* Clearly

$$W = \{(x_1 + x_2, x_3) : (x_1, x_2, x_3) \in \Omega\}.$$

Since  $(x_1, x_2) \in \sigma_T(A_1^* + A_2x_3, A_2^* + A_1x_3)$ ,  $x_1 + x_2$  is an eigenvalue of  $(A_1 + A_2)^* + x_3(A_1 + A_2)$ . Therefore,

$$\begin{aligned}W &= \{(x_1 + x_2, x_3) \in \mathbb{G} : \det[(A_1 + A_2)^* + x_3(A_1 + A_2) - (x_1 + x_2)I] = 0\} \\ &= \{(s, p) \in \mathbb{G} : \det[(A_1 + A_2)^* + p(A_1 + A_2) - sI] = 0\},\end{aligned}$$

where  $\omega(A_1 + A_2) < 1$ , by Lemma 2.6. Therefore, by Theorem 5.4,  $W$  is a distinguished variety in  $\mathbb{G}$ . Also, the existence of a distinguished variety  $V$  in  $\mathbb{D}^2$  with  $\pi(V) = W$  is guaranteed by Lemma 5.3. ■

It is still unknown whether the other way is also true, i.e, whether every distinguished variety in  $\mathbb{G}$  or in  $\mathbb{D}^2$  gives rise to a distinguished variety in  $\mathbb{E}$ . Our wild guess to this question is in the negative direction and the reason is that every distinguished variety in the symmetrized bidisc has representation in terms of the fundamental operator of a  $\Gamma$ -contraction as was shown in [28] and it is still unknown whether every  $\Gamma$ -contraction gives rise to an  $\mathbb{E}$ -contraction although the other way is true according to Theorem 5.2.

## 6. A VON-NEUMANN TYPE INEQUALITY FOR $\mathbb{E}$ -CONTRACTIONS

**Theorem 6.1.** *Let  $\Upsilon = (T_1, T_2, T_3)$  be an  $\mathbb{E}$ -contraction on a Hilbert space  $\mathcal{H}$  such that  $(T_1^*, T_2^*, T_3^*)$  is a pure  $\mathbb{E}$ -contraction and that  $\dim \mathcal{D}_{T_3} < \infty$ . Suppose that the fundamental operators  $A_1, A_2$  of  $(T_1, T_2, T_3)$  satisfy  $[A_1, A_2] = 0$  and  $[A_1^*, A_1] = [A_2^*, A_2]$ . If*

$$\Omega_\Upsilon = \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : (x_1, x_2) \in \sigma_T(A_1^* + x_3A_2, A_2^* + x_3A_1)\},$$

*then for every scalar or matrix-valued polynomial  $p$  in three variables,*

$$\|p(T_1, T_2, T_3)\| \leq \max_{(x_1, x_2, x_3) \in \Omega_\Upsilon \cap b\mathbb{E}} \|p(x_1, x_2, x_3)\|.$$

*Moreover, when  $\omega(A_1 + A_2z) < 1$  for every  $z$  of unit modulus,  $\Omega_\Upsilon \cap \mathbb{E}$  is a distinguished variety in the tetrablock.*



*Proof.* Suppose that  $\dim \mathcal{D}_{T_3} = n$  and then  $A_1, A_2$  are commuting matrices of order  $n$ . We apply Theorem 3.2 to the pure  $\mathbb{E}$ -contraction  $(T_1^*, T_2^*, T_3^*)$  to get an  $\mathbb{E}$ -co-isometric extension on  $H^2(\mathcal{D}_{T_3})$  of  $(T_1, T_2, T_3)$ . Therefore,

$$T_{A_1^* + A_2 z}^*|_{\mathcal{H}} = T_1, \quad T_{A_2^* + A_1 z}^*|_{\mathcal{H}} = T_2, \quad \text{and} \quad T_z^*|_{\mathcal{H}} = T_3.$$

Let  $\varphi$  and  $\psi$  denote the  $\mathcal{L}(\mathcal{D}_{T_3})$  valued functions  $\varphi(z) = A_1^* + A_2 z$  and  $\psi(z) = A_2^* + A_1 z$  respectively. Let  $p$  be a scalar or matrix-valued polynomial in three variables and let  $p_*$  be the polynomial satisfying  $p_*(A, B) = p(A^*, B^*)^*$  for any two commuting operators  $A, B$ . Then

$$\begin{aligned} \|p(T_1, T_2, T_3)\| &\leq \|p(T_\varphi^*, T_\psi^*, T_z^*)\|_{H^2(\mathcal{D}_{T_3})} \\ &= \|p_*(T_\varphi, T_\psi, T_z)\|_{H^2(\mathcal{D}_{T_3})} \\ &\leq \|p_*(M_\varphi, M_\psi, M_z)\|_{L^2(\mathcal{D}_{T_3})} \\ &= \max_{\theta \in [0, 2\pi]} \|p_*(\varphi(e^{i\theta}), \psi(e^{i\theta}), e^{i\theta}I)\|. \end{aligned}$$

It is obvious from Theorem 2.8 that  $(M_\varphi, M_\psi, M_z)$  is an  $\mathbb{E}$ -unitary as  $M_z$  on  $L^2(\mathcal{D}_{T_3})$  is a unitary. Therefore,  $M_\varphi$  and  $M_\psi$  are commuting normal operators and hence  $\varphi(z)$  and  $\psi(z)$  are commuting normal operators for every  $z$  of unit modulus. Therefore,

$$\|p_*(\varphi(e^{i\theta}), \psi(e^{i\theta}), e^{i\theta}I)\| = \max\{|p_*(\lambda_1, \lambda_2, e^{i\theta})| : (\lambda_1, \lambda_2) \in \sigma_T(\varphi(e^{i\theta}), \psi(e^{i\theta}))\}.$$

Let us define

$$\Omega_\Upsilon = \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}$$

and

$$\begin{aligned} \Omega_\Upsilon^* &= \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : (x_1, x_2) \in \sigma_T(A_1 + x_3 A_2^*, A_2 + x_3 A_1^*)\} \\ &= \{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overline{\mathbb{E}} : (x_1, x_2) \in \sigma_T(A_1^* + x_3 A_2, A_2^* + x_3 A_1)\}. \end{aligned}$$

It is obvious that both  $\Omega_\Upsilon$  and  $\Omega_\Upsilon^*$  are one-dimensional subvarieties of  $\overline{\mathbb{E}}$ . We now show that if  $(\lambda_1, \lambda_2) \in \sigma_T(\varphi(e^{i\theta}), \psi(e^{i\theta}))$  then  $(\lambda_1, \lambda_2, e^{i\theta})$  is in  $\overline{\mathbb{E}}$ . There exists a unit vector  $\nu$  such that

$$(A_1^* + e^{i\theta} A_2)\nu = \lambda_1 \nu \quad \text{and} \quad (A_2^* + e^{i\theta} A_1)\nu = \lambda_2 \nu.$$

Taking inner product with  $\nu$  we get  $\beta_1 + \bar{\beta}_2 e^{i\theta} = \lambda_1$  and  $\beta_2 + \bar{\beta}_1 e^{i\theta} = \lambda_2$ , where  $\beta_1 = \langle A_1^* \nu, \nu \rangle$  and  $\beta_2 = \langle A_2^* \nu, \nu \rangle$ . Since  $A_1, A_2$  are the fundamental operators, by Lemma 2.6,  $\omega(A_2^* + z A_1)$  is not greater than 1 for every  $z$  of unit modulus. Therefore,

$$|\lambda_2| = |\beta_1 + \bar{\beta}_2 e^{i\theta}| \leq 1.$$

Again

$$\bar{\lambda}_2 e^{i\theta} = \beta_1 + \bar{\beta}_2 e^{i\theta} = \lambda_1.$$

Therefore by (1.1),  $(\lambda_1, \lambda_2, e^{i\theta}) \in b\mathbb{E} \subseteq \overline{\mathbb{E}}$ . Therefore, we conclude that

$$\begin{aligned} \|p(T_1, T_2, T_3)\| &\leq \max_{(x_1, x_2, x_3) \in \Omega_\Upsilon^* \cap b\mathbb{E}} \|p_*(x_1, x_2, x_3)\| \\ &= \max_{(x_1, x_2, x_3) \in \Omega_\Upsilon \cap b\mathbb{E}} \|p(x_1, x_2, x_3)\|. \end{aligned}$$

If  $\omega(A_1 + A_2 z) < 1$  for all  $z \in \mathbb{T}$ , by Lemma 2.6,  $\omega(A_1^* + A_2 z) = \|A_1^* + A_2 z\|_{\infty, \mathbb{T}} < 1$ . It is now obvious from Theorem 4.5 that  $\Omega_\Upsilon \cap \mathbb{E}$  is a distinguished variety in the tetrablock. ■

We have established the fact that if an  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  satisfies the hypotheses of Theorem 6.1 then there is an  $\mathbb{E}$ -co-isometric extension of  $(T_1, T_2, T_3)$  that lives on the corresponding variety  $\Omega_\Upsilon$ . This also makes  $\Omega_\Upsilon^*$  a complete spectral set for  $(T_1^*, T_2^*, T_3^*)$ . Obviously this is valid when  $(T_1, T_2, T_3)$  consists of matrices and  $\|T_3\| < 1$ . We do not know whether there is a bigger class of  $\mathbb{E}$ -contractions for which von-Neumann type inequality is valid on such a one-dimensional subvariety. This line of proof of Theorem 6.1 will no longer be valid if there is a bigger class and also  $\Omega_\Upsilon^*$  will not be a complete spectral set for  $(T_1^*, T_2^*, T_3^*)$  in that case as we have used the functional model of  $(T_1^*, T_2^*, T_3^*)$  in the proof.

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